Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00036-9

J. Appl. Maths Mechs, Vol. 66, No. 2, pp. 297–305, 2002

0021-8928/02/\$—see front matter

DOUBLY PERIODIC CONTACT PROBLEMS FOR AN ELASTIC LAYER†

V. M. ALEKSANDROV

Moscow

(*Received* 3 *May* 2001)

The problem of the indentation of a doubly periodic system of punches into an elastic layer is considered. For a relatively thick layer and relatively large values of the semiperiods, an asymptotic solution is obtained. As an example, the cases of a doubly periodic system of plane, inclined and elliptical (in plan) punches and a doubly periodic system of parabolic punches are studied.

Similar problems of the action of a single punch on an elastic layer were considered in [1–3].

1. THE EQUILIBRIUM OF AN ELASTIC LAYER UNDER THE ACTION OF A DOUBLY PERIODIC NORMAL LOAD

Consider an elastic layer of thickness h with mechanical characteristics G (shear modulus) and σ (Poisson's ratio) that lies without friction on a rigid base (Problem 1) or is rigidly clamped along the base (Problem 2). The layer occupies a region $0 \leqslant z \leqslant h$, $|x| < \infty$, $|y| < \infty$, and its upper face $z = h$ is loaded by a doubly periodic pressure $q(x - 2km, y - 2ln)$, where m and n are the lengths of the semiperiods along the x and y axes respectively; $k, l = -\infty, \ldots, -1, 0, 1, \ldots, \infty$. There are no shear loads on the upper face of the layer.

With these assumptions, the normal displacement of points of the upper face of the layer caused by the pressure acting there can be represented in the following form [1, 2]

$$w(x,y,h) = -\frac{1}{4\pi^2\theta h}\int_{\Omega_*} q(\xi,\eta)d\Omega \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{L(r)}{r}\sum_{k=-\infty}^{\infty}\cos\left(s\frac{\xi+2km-x}{h}\right)\times$$

$$\times\sum_{l=-\infty}^{\infty}\cos\left(t\frac{\eta+2ln-y}{h}\right)dsdt \tag{1.1}$$

$$\theta = G(1-\sigma)^{-1},\quad r=\sqrt{s^2+t^2},\quad \Omega_* - \text{region } |\xi|\leqslant m,\ |\eta|\leqslant n$$

$$L(r) = \frac{\operatorname{ch}2r-1}{\operatorname{sh}2r+2r} \quad \text{for Problem 1}$$

$$L(r) = \frac{2\varkappa\operatorname{sh}2r-4r}{2\varkappa\operatorname{ch}2r+1+\varkappa^2+4r^2}\quad (\varkappa = 3-4\sigma)\quad \text{for Problem 2}$$

We will split the function $w(x, y, h)$ of the form (1.1) into two terms in the following way

$$w(x,y,h) = w_1(x,y,h) + w_2(x,y,h) \tag{1.2}$$

$$w_1(x,y,h) = -\frac{1}{4\pi^2\theta h}\int_{\Omega_*} q(\xi,\eta)d\Omega\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1-e^{-2r}}{r}\cos\left(s\frac{\xi+2km-x}{h}\right)\times$$

$$\times\cos\left(t\frac{\eta+2ln-y}{h}\right)dsdt$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 307–315, 2002.

$$w_2(x,y,h)=\frac{1}{4\pi^2\theta h}\int_{\Omega_*} q(\xi,\eta)d\Omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(r)\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\cos\left(s\frac{\xi+2km-x}{h}\right)\times$$
$$\times\cos\left(t\frac{\eta+2ln-y}{h}\right)dsdt$$
$$M(r)=[1-e^{-2r}-L(r)]r^{-1}$$

and simplify them.

Using the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-ar}}{r}\cos sz\cos t\zeta dsdt=\frac{2\pi}{\sqrt{a^2+R^2}}\quad (R=\sqrt{z^2+\zeta^2})$$

to evaluate which the formulae 8.511(4) and 6.611(1) from [4] were employed, we represent the function $w_1(x,y,h)$ in the form

$$w_1(x,y,h)=-\frac{1}{2\pi\theta}\int_{\Omega_*} q(\xi,\eta)\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} Q_{kl}d\Omega \tag{1.3}$$

$$Q_{kl}=R_{kl}^{-1}-(4h^2+R_{kl}^2)^{-1/2},\quad R_{kl}=[(\xi+2km-x)^2+(\eta+2ln-y)^2]^{1/2}$$

Using the transformation

$$\sum_{k=-\infty}^{\infty}\cos\left(s\frac{z+2km}{h}\right)=\cos\left(s\frac{z}{h}\right)\left[1+2\sum_{k=1}^{\infty}\cos\left(s\frac{2km}{h}\right)\right]=2\pi\cos\left(s\frac{z}{h}\right)\sum_{k=-\infty}^{\infty}\delta\left(\frac{2sm}{h}-2\pi k\right)$$

where $\delta(x)$ is the delta function, and where the formula connecting an infinite series of cosines with an infinite series of delta functions [5, p. 49] is taken into account, we represent the function $w_2(x,y,h)$ in the form

$$w_2(x,y,h)=\frac{h}{4mn\theta}\int_{\Omega_*} q(\xi,\eta)\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} M(r_{kl})\cos\frac{\pi k(\xi-x)}{m}\cos\frac{\pi l(\eta-y)}{n}d\Omega,$$

$$r_{kl}=\pi h\left[\left(\frac{k}{m}\right)^2+\left(\frac{l}{n}\right)^2\right]^{1/2} \tag{1.4}$$

Substituting expressions (1.3) and (1.4) into equality (1.2), we obtain

$$w(x,y,h)=-\frac{1}{2\pi\theta}\int_{\Omega_*} q(\xi,\eta)(I_{00}+I_{*0}+I_{0*}+I_{**})d\Omega+\frac{h}{4mn\theta}\int_{\Omega_*} q(\xi,\eta)\times$$
$$\times(J_{00}+2J_{*0}+2J_{0*}+4J_{**})d\Omega \tag{1.5}$$

$$I_{00}=Q_{00},\quad I_{*0}=\sum_{k=1}^{\infty}(Q_{k0}+Q_{-k,0}),\quad I_{0*}=\sum_{l=1}^{\infty}(Q_{0l}+Q_{0,-l})$$

$$I_{**}=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(Q_{kl}+Q_{-k,l}+Q_{k,-l}+Q_{-k,-l})$$

$$J_{00}=\begin{cases}3/2 & \text{for Problem 1}\\ 2-4(\varkappa-1)/(\varkappa+1)^2 & \text{for Problem 2}\end{cases}$$

$$J_{*0}=\sum_{k=1}^{\infty}M(r_{k0})\cos\frac{\pi k(\xi-x)}{m},\quad J_{0*}=\sum_{l=1}^{\infty}M(r_{0l})\cos\frac{\pi l(\eta-y)}{n}$$

$$J_{**}=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}M(r_{kl})\cos\frac{\pi k(\xi-x)}{m}\cos\frac{\pi l(\eta-y)}{n}$$

The series I_{*0}, I_{0*} and I_{**} converge when $|\xi - x| \leqslant C$ and $|\eta - y| \leqslant C$, where C is an arbitrary finite constant. The series J_{*0}, J_{0*} and J_{**} converge under the same conditions, since

$$L(r) = 1 + O(e^{-2r}) \quad (r \to \infty) \tag{1.6}$$

2. FORMULATION OF THE DOUBLY PERIODIC CONTACT PROBLEM FOR AN ELASTIC LAYER

Suppose a doubly periodic system of identical rigid punches is impressed into the upper face of the layer $z = h$. There are no friction forces between the punches and the surface of the layer, and an identical system of forces acts on each punch: a force P and moments M_x and M_y. Then the areas of contact of all the punches with the surface of the layer will be identical. Suppose Ω is the area of contact of the punch in zone $|x| \leqslant m$, $|y| \leqslant n$, and suppose $\delta(x, y)$ is the deflection of points of the surface of the layer under this punch (there will be similar deflections of the surface of the layer under all the other punches).

By satisfying the condition of contact of the selected punch with the surface of the layer

$$w(x, y, h) = -\delta(x, y) \quad ((x, y) \in \Omega) \tag{2.1}$$

and using formula (1.5), we arrive at the following two-dimensional integral equation of the first kind for determining the contact pressure $q(x, y)$

$$\int_\Omega \frac{q(\xi,\eta)}{R_{00}} d\Omega = 2\pi\theta\delta(x, y) + \int_\Omega q(\xi,\eta) F(\xi - x, \eta - y) d\Omega \tag{2.2}$$

$$F(z,\zeta) = \frac{1}{\sqrt{4h^2 + R_{00}^2}} - I_{*0} - I_{0*} - I_{**} + \frac{\pi h}{2mn}(J_{00} + 2J_{*0} + 2J_{0*} + 4J_{**})$$

$$(x, y) \in \Omega, \quad z = \xi - x, \quad \zeta = \eta - y$$

Note that $F(\xi - x, \eta - y)$ is the regular part of integral equation (2.2). Equation (2.2) must be supplemented by the conditions of equilibrium of the punch

$$P = \int_\Omega q(\xi,\eta) d\Omega, \quad M_x = \int_\Omega q(\xi,\eta)\eta d\Omega, \quad M_y = \int_\Omega q(\xi,\eta)\xi d\Omega \tag{2.3}$$

which can be used to determine the displacement $\delta + \alpha x + \beta y$ of the punch as a rigid whole, which enters as a term into the function $\delta(x, y)$. The remaining part of the function $\delta(x, y)$ is defined by the shape of the punch base.

If the region Ω is unknown in advance, it is still necessary to require that, on possible variations of Ω, the solution of Eq. (2.2) yields a minimum of the functional [2, 6]

$$I = \int_\Omega q(\xi,\eta)\delta(\xi,\eta) d\Omega \tag{2.4}$$

This condition is often replaced by

$$q(x, y) = 0, \quad (x, y) \in L \tag{2.5}$$

where L is the contour of the region Ω, which follows [7] from the condition for a minimum of functional (2.4) but is not in the general case sufficient to determine the contact area Ω.

3. ASYMPTOTIC SIMPLIFICATION OF THE REGULAR PART OF THE KERNEL OF THE INTEGRAL EQUATION

Suppose $h \sim m \sim n$ and $a = \max R_{00}$ in Ω. We will assume $\lambda = h/a$, $\mu = m/a$ and $\nu = n/a$ to be the major parameters. Taking this into account, we will expand the first four components in the expression for the function $F(z, \zeta)$ on the right-hand side of Eq. (2.2) in series. The first component will be expanded in terms of the parameter λ^{-1}. For the component I_{*0} in the expressions for Q_{k0} and $Q_{k,0}$ the first terms will be expanded in terms of μ^{-1}, and the second terms will be expanded in terms of the parameter $(\lambda^2+\mu^2)^{-1/2}$. For the component I_{0*} in the expressions for Q_{0l} and $Q_{0,-l}$ the first terms will be expanded in terms of the parameter ν^{-1}, and the second terms will be expanded in terms of the parameter $(\lambda^2+\nu^2)^{-1/2}$. For the component I_{**} in the expressions for Q_{kl}, Q_{-kl}, $Q_{k,-l}$ and $Q_{-k,-l}$ the first terms will be expanded in terms of the parameters $(\mu^2+\nu^2)^{-1/2}$, and the second terms will be expanded in terms of the parameter $(\lambda^2+\mu^2+\nu^2)^{-1/2}$. The series obtained will converge when $|\xi - x| \leqslant 2a$ and $|\eta - y| \leqslant 2a$ if the following conditions are satisfied

$$\lambda > 1, \qquad \mu > 1+\sqrt{2}, \qquad \nu > 1+\sqrt{2} \tag{3.1}$$

We will now deal with the expansion in series in the parameters μ^{-1} and ν^{-1} of the terms J_{*0}, J_{0*} and J_{**} in the expression for the function $F(z, \zeta)$. We will represent J_{*0} in the form

$$J_{*0} = \sum_{j=0}^{\infty} \frac{(-1)^j A_j}{(2j)!}\left[\frac{\pi(\xi - x)}{m}\right]^{2j}, \quad A_j = \sum_{k=1}^{\infty} M(r_{k0})k^{2j} \tag{3.2}$$

Taking (1.6) into account, we will have

$$A_j = O\left(\sum_{k=1}^{\infty} \tilde{x}^k k^{2j}\right), \quad \tilde{x} = \exp\left(-\frac{2\pi h}{m}\right) < 1 \tag{3.3}$$

The series under the order sign in expression (3.3) for A_j is easily summed [8] for each specific value of $j \geqslant 0$; however, to evaluate A_j, it is more convenient to use the inequality

$$\sum_{k=1}^{\infty} \tilde{x}^k k^n \leqslant \frac{\tilde{x} n!}{(1-\tilde{x})^{n+1}} \tag{3.4}$$

We then have $A_j = O[(2j)!(1-\tilde{x})^{-2j}]$ and it can be seen that series (3.2) for J_{*0} converges when $|\xi - x| \leqslant 2a$ if the following condition is satisfied

$$\frac{2\pi}{\mu}\left[1 - \exp\left(-\frac{2\pi}{m_1}\right)\right]^{-1} \leqslant 1; \quad m_1 = \frac{\mu}{\lambda} = \frac{m}{h} \sim 1 \tag{3.5}$$

Similarly, we find that the expansion

$$J_{0*} = \sum_{s=0}^{\infty} \frac{(-1)^s B_s}{(2s)!}\left[\frac{\pi(\eta - y)}{n}\right]^{2s}, \quad B_s = \sum_{l=1}^{\infty} M(r_{0j})l^{2s} \tag{3.6}$$

will converge when $|\eta - y| \leqslant 2a$ if the following condition is satisfied

$$\frac{2\pi}{\nu}\left[1 - \exp\left(-\frac{2\pi}{n_1}\right)\right]^{-1} \leqslant 1; \quad n_1 = \frac{\nu}{\lambda} = \frac{n}{h} \sim 1 \tag{3.7}$$

Finally, consider the expansion for J_{**}

$$J_{**} = \sum_{j=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{j+s} C_{js}}{(2j)!(2s)!}\left[\frac{\pi(\xi - x)}{m}\right]^{2j}\left[\frac{\pi(\eta - y)}{n}\right]^{2s} \tag{3.8}$$

$$C_{js} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} M(r_{kl})k^{2j}l^{2s}$$

Again using (1.6), inequality (3.4) and the inequality

$$\sqrt{2(k^2+l^2)} \geqslant k + l; \quad k \geqslant 1, \quad l \geqslant 1 \tag{3.9}$$

we estimate the quantities C_{js}, and as a result we find that expansion (3.8) will converge when $|\xi - x| \leqslant 2a$ and $|\eta - y| \leqslant 2a$ if the following conditions are satisfied

$$\frac{2\pi}{\mu}\left[1-\exp\left(-\frac{\sqrt{2}\pi}{m_1}\right)\right]^{-1} < 1,\quad \frac{2\pi}{\nu}\left[1-\exp\left(-\frac{\sqrt{2}\pi}{n_1}\right)\right]^{-1} < 1 \tag{3.10}$$

Note that conditions (3.10) are stronger than conditions (3.5) and (3.7), and stronger than the second and third conditions of (3.1).

Thus, if the first condition of (3.1) and conditions (3.10), which impose constraints on the parameters λ, μ and ν are satisfied, then the regular part of the kernel of the integral of Eq. (2.2) can be represented in the form

$$F(z,\zeta) = \sum_{j=0}^{\infty}\sum_{s=0}^{\infty} D_{js} z^{2j}\zeta^{2s} \tag{3.11}$$

Further, in this expansion, we will confine ourselves to retaining the terms $z^0\zeta^0$, $z^2\zeta^0$ and $z^0\zeta^2$, where the coefficients D_{00}, D_{10} and D_{01} will have the form

$$\begin{aligned}
&D_{00} = \frac{1}{h}\left[\frac{1}{2} - 2S_{*0}^{(0,0)} - 2S_{0*}^{(0,0)} - 4S_{**}^{(0,0)} + \frac{\pi}{2m_1n_1}\left(2 - A + 2T_{*0}^{(0,0)} + 2T_{0*}^{(0,0)} + 4T_{**}^{(0,0)}\right)\right]\\
&S_{*0}^{(0,0)} = \sum_{k=1}^{\infty}\left(\frac{1}{g_{k0}} - \frac{1}{p_{k0}}\right),\quad S_{0*}^{(0,0)} = \sum_{l=1}^{\infty}\left(\frac{1}{g_{0l}} - \frac{1}{p_{0l}}\right)\\
&S_{**}^{(0,0)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{g_{kl}} - \frac{1}{p_{kl}}\right)\\
&T_{*0}^{(0,0)} = \sum_{k=1}^{\infty} M(r_{k0}),\quad T_{0*}^{(0,0)} = \sum_{l=1}^{\infty} M(r_{0l}),\quad T_{**}^{(0,0)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M(r_{kl})\\
&D_{10} = \frac{1}{h^3}\left[-\frac{1}{16} - S_{*0}^{(1,0)} + S_{0*}^{(1,0)} + 2S_{**}^{(1,0)} - \frac{\pi}{2m_1n_1}\left(T_{*0}^{(1,0)} + 2T_{**}^{(1,0)}\right)\right]\\
&S_{*0}^{(1,0)} = \sum_{k=1}^{\infty}\left(\frac{2}{g_{k0}^3} + \frac{1}{p_{k0}^3} - \frac{3g_{k0}^2}{p_{k0}^5}\right),\quad S_{0*}^{(1,0)} = \sum_{l=1}^{\infty}\left(\frac{1}{g_{0l}^3} - \frac{1}{p_{0l}^3}\right)\\
&S_{**}^{(1,0)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{g_{kl}^3} - \frac{3g_{k0}^2}{g_{kl}^5} - \frac{1}{p_{kl}^3} + \frac{3g_{k0}^2}{p_{kl}^5}\right)\\
&T_{*0}^{(1,0)} = \sum_{k=1}^{\infty} M(r_{k0})\frac{\pi^2k^2}{m_1^2},\quad T_{**}^{(1,0)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M(r_{kl})\frac{\pi^2k^2}{m_1^2}\\
&D_{01} = \frac{1}{h^3}\left[-\frac{1}{16} + S_{*0}^{(0,1)} - S_{0*}^{(0,1)} + 2S_{**}^{(0,1)} - \frac{\pi}{2m_1n_1}\left(T_{0*}^{(0,1)} + 2T_{**}^{(0,1)}\right)\right]\\
&S_{*0}^{(0,1)} = \sum_{k=1}^{\infty}\left(\frac{1}{g_{k0}^3} - \frac{1}{p_{k0}^3}\right),\quad S_{0*}^{(0,1)} = \sum_{l=1}^{\infty}\left(\frac{2}{g_{0l}^3} + \frac{1}{p_{0l}^3} - \frac{3g_{0l}^2}{p_{0l}^5}\right)\\
&S_{**}^{(0,1)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{g_{kl}^3} - \frac{3g_{0l}^2}{g_{kl}^5} - \frac{1}{p_{kl}^3} + \frac{3g_{0l}^2}{p_{kl}^5}\right)\\
&T_{0*}^{(0,1)} = \sum_{l=1}^{\infty} M(r_{0l})\frac{\pi^2l^2}{n_1^2},\quad T_{**}^{(0,1)} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} M(r_{kl})\frac{\pi^2l^2}{n_1^2}\\
&g_{kl} = 2\sqrt{k^2m_1^2 + l^2n_1^2},\quad p_{kl} = 2\sqrt{1 + k^2m_1^2 + l^2n_1^2}\\
&r_{kl} = \pi\sqrt{k^2m_1^{-2} + l^2n_1^{-2}}
\end{aligned} \tag{3.12}$$

If $m_1 = n_1$, then $S_{*0}^{(0,0)} = S_{0*}^{(0,0)}$, $T_{*0}^{(0,0)} = T_{0*}^{(0,0)}$, $S_{*0}^{(1,0)} = S_{0*}^{(0,1)}$, $S_{*0}^{(1,0)} = S_{*0}^{(0,1)}$, $S_{**}^{(1,0)} = S_{**}^{(0,1)}$, $T_{*0}^{(1,0)} = T_{0*}^{(0,1)}$, $T_{**}^{(1,0)} = T_{**}^{(0,1)}$ and $D_{10} = D_{01}$.

4. DOUBLY PERIODIC SYSTEM OF PLANE INCLINED PUNCHES THAT ARE ELLIPTICAL IN PLAN

Let region Ω be an ellipse with semiaxes a and b ($a > b$) oriented along the x and y axes respectively. For simplicity, we will assume that $m_1 = n_1$, and put $a_0 = hD_{00}$, $a_1 = h^3D_{10} = h^3D_{01}$. Substituting expansion (3.11), where we retain only terms with coefficients D_{00}, D_{10} and D_{01}, into Eq. (2.2), we obtain the integral equation

$$\int_\Omega q(\xi,\eta)\frac{d\Omega}{\sqrt{(\xi-x)^2+(\eta-y)^2}} = 2\pi\theta\delta(x,y) + \frac{1}{h}\int_\Omega q(\xi,\eta)\left\{a_0 + \frac{a_1}{h^2}[(\xi-x)^2+(\eta-y)^2]\right\}d\Omega,$$
$$(x,y)\in\Omega \tag{4.1}$$

Here, $\delta(x,y) = \delta + \alpha x + \beta y$, where δ is the translated displacement of the selected punch in the negative direction of the z axis, and α and β are its angles of rotation about the y and x axes.

By virtue of the well-known Galin theorem [9] concerning the form of the solution of the contact problem for a half-space in the case of a punch that is elliptical in plan, the solution of integral equation (4.1) must be sought in the form

$$q(x,y) = \theta\left(a_{00} + a_{10}\frac{x}{a} + a_{01}\frac{y}{a} + a_{20}\frac{x^2}{a^2} + a_{02}\frac{y^2}{a^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{-1/2} \tag{4.2}$$

Substituting expression (4.2) into Eq. (4.1), and evaluating the integrals on the left and right [1, 2], we obtain the following system of linear algebraic equations for determining the coefficients a_{00}, a_{10}, a_{01}, a_{20} and a_{02}

$$\begin{aligned}
&a_{00}S_{00} + \frac{1}{2}a_{20}S_{01}\varepsilon^2 + \frac{1}{2}a_{02}S_{10}\varepsilon^2 = \frac{\delta}{b} + \frac{1}{\lambda}\left(a_{00}A_{00} + \frac{1}{3}a_{20}A_{20} + \frac{1}{3}a_{02}\varepsilon^2A_{02}\right)\\
&a_{20}\left(S_{20} - \frac{1}{2}S_{11}\varepsilon^2\right) + a_{02}\varepsilon^2\left(S_{11}\varepsilon^2 - \frac{1}{2}S_{20}\right) = \frac{1}{\lambda}B\left(a_{00} + \frac{1}{3}a_{20} + \frac{1}{3}a_{02}\varepsilon^2\right)\\
&a_{20}\left(S_{11} - \frac{1}{2}S_{02}\varepsilon^2\right) + a_{02}\varepsilon^2\left(S_{02}\varepsilon^2 - \frac{1}{2}S_{11}\right) = \frac{1}{\lambda}B\left(a_{00} + \frac{1}{3}a_{20} + \frac{1}{3}a_{02}\varepsilon^2\right)\\
&a_{10}S_{10} = \frac{\alpha}{\varepsilon} - \frac{2}{3\lambda}a_{10}B, \quad a_{01}S_{01} = \frac{\beta}{\varepsilon} - \frac{2}{3\lambda}a_{01}\varepsilon^2B
\end{aligned} \tag{4.3}$$

Here, we have introduced the following notation

$$\begin{aligned}
&S_{00} = \mathbf{K}(e), \quad S_{10} = \frac{\mathbf{K}(e) - \mathbf{E}(e)}{e^2}, \quad S_{01} = \frac{\mathbf{E}(e) - \varepsilon^2\mathbf{K}(e)}{e^2\varepsilon^2}; \quad e^2 = 1 - \frac{b^2}{a^2}\\
&S_{20} = \frac{-2(2-\varepsilon^2)\mathbf{E}(e) + (3-\varepsilon^2)\mathbf{K}(e)}{3e^4}, \quad S_{11} = \frac{(1+\varepsilon^2)\mathbf{E}(e) - 2\varepsilon^2\mathbf{K}(e)}{3e^4\varepsilon^2}\\
&S_{02} = \frac{-2(2\varepsilon^2-1)\mathbf{E}(e) + \varepsilon^2(3\varepsilon^2-1)\mathbf{K}(e)}{3e^4\varepsilon^4}\\
&\varepsilon^2 = 1 - e^2, \quad B = a_1\lambda^{-2}, \quad A_{00} = a_0 + B(1+\varepsilon^2)/3\\
&A_{20} = a_0 + B(3+\varepsilon^2)/5, \quad A_{02} = a_0 + B(1+3\varepsilon^2)/5
\end{aligned} \tag{4.4}$$

where $\mathbf{K}(e)$ and $\mathbf{E}(e)$ are complete elliptical integrals.

Table 1

e	$\lambda = 2$	4	6	8	10
		Problem 1			
0.6	5.242	4.325	4.056	3.930	3.857
	(5.147)	(4.287)	(4.032)	(3.913)	(3.844)
0.8	4.384	3.705	3.503	3.409	3.354
	(4.315)	(3.677)	(3.486)	(3.396)	(3.344)
		Problem 2			
0.6	5.619	4.478	4.147	3.994	3.907
	(5.477)	(4.433)	(4.122)	(3.997)	(3.894)
0.8	4.659	3.817	3.572	3.457	3.392
	(4.562)	(3.785)	(3.553)	(3.444)	(3.381)

After solving system of equations (4.3), we obtain the contact pressure by means of formula (4.2), and then the forces and moments acting on the punch by means of formulae (2.3). These take the form

$$P = 2\pi ab\theta\left[a_{00} + \frac{1}{3}(a_{20} + \varepsilon^2 a_{02})\right]$$

$$M_y = \frac{2\pi a^2 b\theta}{3} a_{10} = \frac{2\pi a^2 b\theta\alpha}{3\varepsilon}\left(S_{10} + \frac{2B}{3\lambda}\right)^{-1} \qquad (4.5)$$

$$M_x = \frac{2\pi a^2 b\varepsilon^2\theta}{3} a_{01} = \frac{2\pi a^2 b\varepsilon\theta\beta}{3}\left(S_{01} + \frac{2\varepsilon^2 B}{3\lambda}\right)^{-1}$$

Let $m_1 = n_1 = 1$, i.e. $\lambda = \mu = \nu$. Then all the estimates (3.1), (3.5), (3.7) and (3.10) will reduce to the condition $\lambda > 6.358$. However, this condition is overstated, and in fact, as shown by calculations, formulae (4.2), (4.3) and (4.5) can be used when $\lambda \geqslant 2$. For the problems considered, from formulae (3.12) we find

$$a_0 = 1.2220\ (1.1676), \quad a_1 = -0.43395\ (-0.39538) \quad \text{for Problem 1}$$

$$a_0 = 1.4323\ (1.3768), \quad a_1 = -0.61770\ (-0.62755) \quad \text{for Problem 2}$$

For Problem 2 it is assumed that $\sigma = 0.3$. The values of the quantities a_0 and a_1 for a single punch on the layer are given in parentheses.

Table 1 gives values of $P/(a\theta\delta)$ calculated by means of the first formula of (4.5) for eccentricities $e = 0.6$ and 0.8 and different values of λ. The corresponding values for a single punch on the layer are given in parenthesis [1, 2].

5. A DOUBLY PERIODIC SYSTEM OF PARABOLIC PUNCHES

Again, we will start from integral equation (4.1), in which now $\delta(x, y) = \delta - x^2/(2R_1) - y^2/(2R_2)$, where R_1 and R_2 are the radii of curvature at the tip of the selected parabolic punch, and δ is the penetration of this punch.

From the above-mentioned Galin theorem and condition (2.5) it follows that, in this case, the solution of integral equation (4.1) must be sought in the form

$$q(x,y) = a_{00}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} \qquad (5.1)$$

where a and b are the as yet unknown semiaxes of the elliptical area of contact.

Substituting (5.1) into (4.1), and evaluating the integrals on the left and right [1–3], we obtain three relations

$$a_{00}=\frac{2\theta\delta}{bA}\left(A=\mathbf{K}(e)-\frac{2}{3\lambda}\left(a_0+\frac{B(1+\varepsilon^2)}{5}\right)\right)$$
$$\frac{1}{R_1}=\frac{ba_{00}}{a^2\theta}\left(S_{10}+\frac{2B}{3\lambda}\right),\quad \frac{1}{R_2}=\frac{ba_{00}}{a^2\theta}\left(S_{01}+\frac{2B}{3\lambda}\right) \tag{5.2}$$

We add the further relation

$$a_{00}=3P/(2\pi ab) \tag{5.3}$$

obtained using the first formula of (2.3) and formula (5.1).

Comparing the first formula of (5.2) with (5.3), we determine the relation between the deflection δ for the selected punch and the impressing force P

$$\delta=3PA/(4\pi\theta a) \tag{5.4}$$

Substituting expression (5.3) into the second formula of (5.2), we find

$$a^3=\frac{3Ph^3R_1S_{10}}{2(\pi\theta h^3-Pa_1R_1)} \tag{5.5}$$

Substituting expression (5.3) and then (5.5) into the third formula of (5.2), we obtain

$$\frac{S_{01}}{S_{10}}=N,\quad N=\frac{R_1\cdot\pi\theta h^3-Pa_1R_2}{R_2\cdot\pi\theta h^3-Pa_1R_1} \tag{5.6}$$

It can be seen that $N>1$ when $R_1>R_2$.

Figure 1 shows a graph of e against N, determined by the first formula of (5.6) (earlier [3], this graph was given with a considerable error for e values close to unity).

Following the results obtained earlier [3], the calculation can now be carried out using the following scheme. Knowing the initial parameters of the problem G, σ, h, R_1, R_2, a_0, a_1 and P, from the second formula of (5.6) we calculate the value of N, and from the graph in the figure, or from the first formula of (5.6), we determine the corresponding value of the eccentricity e. Then, from formula (5.5) we find the semiaxis a of the elliptical area of contact, we check that $\lambda=h/a\geqslant 2$, and we then find the semiaxis $b=a\sqrt{1-e^2}$. From formula (5.4), we determine the sag of the punch δ, from formula (5.3) the value of a_{00} and, finally, from formula (5.1) the distribution function of the contact pressures.

From example, for Problem 2, for $m_1=n_1=1$, and also for $h=0.07$ m, $R_1=2$ m, $R_2=1$ m, $P=0.1\times10^7$ N and $E=0.98\times10^{11}$ N/m^2, $\sigma=0.35$ (brass, $G=2E(1+\sigma)$), we obtain

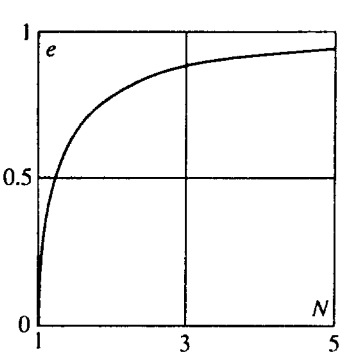

Fig. 1

$$N = 1.9799\ (1.9796),\quad e = 0.77265\ (0.77260)$$
$$a = 0.026275\text{ m}\ (0.026272\text{ m}),\quad \lambda = 2.6641(2.6644)$$
$$b = 0.016680\text{ m}\ (0.016680\text{ m}),\quad \delta = 0.25936\text{ mm}\ (0.26166\text{ mm})$$
$$a_{00} = 0.10894 \times 10^{10}\text{ N/m}^2 \quad (0.10896 \times 10^{10}\text{ N/m}^2)$$

Values of the quantities found for a single punch on the layer are given in parenthesis.

Remark. If in integral equation (2.2) the region Ω is symmetrical about the x and y axes, while the functional $\delta(x, y)$ is even in x and y, then Eq. (2.2) also describes the contact problem for a rectangular parallelepiped $0 \leqslant z \leqslant h$, $|x| \leqslant m$, $|y| \leqslant n$, the side faces of which are hinged, and the lower face of which is hinged (Problem 1) or rigidly clamped (Problem 2).

I wish to thank I. I. Argatov and I. A. Soldatenkov for discussing certain aspects for the work, and A. I. Zhurov for help with the calculations.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00038).

REFERENCES

1. VOROVICH, I. I., ALEKSANDROV, V. M. and BABESHKO, V. A., *Non-classical Mixed Problems of Elasticity Theory*. Nauka, Moscow, 1974.
2. ALEKSANDROV, V. M. and POZHARSKII, D. A., *Non-classical Three-dimensional Problems of the Mechanics of Contact Interactions of Elastic Bodies*. Faktorial, Moscow, 1998.
3. ALEKSANDROV, V. M. and SHMATKOVA, A. A., The indentation of a parabolic punch into an elastic layer and of two parabolic punches into an elastic half-space. *Izv. Ross. Akad. Nauk. MTT*, 1998, 4, 149–155.
4. GRADSHTEYN, I. S. and RYZHIK, I. M., *Tables of Integrals, Sums, Series and Products*. Academic Press, New York, 1980.
5. GEL'FAND, I. M. and SHILOV, G. Ye., *Generalized Functions and Actions On Them*. Fizmatgiz, Moscow, 1959.
6. GALIN, L. A. (Ed.), *The Development of the Theory of Contact Problems in the USSR*. Nauka, Moscow, 1976.
7. BARENBLATT, G. I., Conditions of infinity in continuum mechanics. Static problems of elasticity theory. *Prikl. Mat. Mekh.*, 1960, **24**, 2, 316–322.
8. PRUDNIKOV, A. P., BRYCHKOV, Yu. A. and MARICHEV, O. I., *Integrals and Series*, Vol. 1, *Elementary Functions*. Gordon and Breach Sci. Publ., New York, 1986.
9. GALIN, L. A., *Contact Problems of Elasticity and Viscoelasticity Theory*. Nauka, Moscow, 1980.

Translated by P.S.C.